Queuing theory and teletraffic systems

Lecture 2
Stochastic processes, Poisson Process, Markov chains

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Outline

• Stochastic processes
• Poisson process
• Markov process
• Markov chains
  – Discrete time Markov chains
  – Continuous time Markov chains
• Transient solution
• Stationary solution
• Balance equations
Lecture 1 review

• Queuing theory: performance evaluation of resource sharing system
• Block diagram of queuing system

![Block diagram of queuing system]

- Arrival
- Buffer/queue
- Exit system
- Server
- Interrupted service
- Finished task
- Blocking
Description of queuing system

• System parameters
  – Number of servers (tasks processed in parallel)
  – Buffer capacity
    • Infinite (no jobs will be blocked)
    • Finite (some jobs will be blocked)
  – Order of service (FIFO, priority, random)

• Service demand (stochastic, given by probability distributions)
  – Arrival process: how customers arrive to the system
  – Service process: how much service a customer demands
Performance measures

• Number of customers in the system (N)
  – Number of customers in queue (N_q)
  – Number of customers in server (N_s)
• System time
  – Waiting time (W)
  – Service time (x)
• Probability of blocking
• Utilization of the server
• Transient measures
  – How will the system change in the near future
• Stationary measures
  – How does the system behave in the long run
  – Average measure
  – We will encounter these kind of measures in this course
Stochastic process

• **Stochastic process / random process** \( \{ X_t \text{ or } X(t) \} \)
  
  – A system that evolves in time (e.g. length of queue, temperature of a city)
  
  – Family of random variables
  
  – Realization: \( X_t(e) \) associated with a given value ‘e’
    
    • Example: \( X_t(1) = x_0 \) or \( X(t_0) = x_0 \)
  
  – State space: \( S = \{ x_0, x_1, \ldots \} \)
  
  – Parameter space: set of all values of \( t \)
  
  – Both state space and parameter space could be discrete or continuous
  
  – Depending on parameter space, we can classify stochastic processes as **discrete-time** or **continuous-time** stochastic process
  
  – Discrete-time stochastic processes are also called **random sequences**
Stochastic process (contd.)

• Important quantities
  
  – **Time-dependant distribution**: probability that the stochastic process $X(t)$ takes a value in particular subset of $S$ at a given instant $t$
  
  – **Stationary distribution**: probability that the stochastic process $X(t)$ takes a value in particular subset of $S$ as $t \to \infty$
  
  – **Hitting probability**: probability that a given state in $S$ will ever be entered
  
  – **First passage time**: time when the stochastic process first enters a given state or set of states starting from a given initial state
  
  – **Covariance and correlation**: defines the relation between two stochastic processes ($X_t$ and $X_s$) for different times $s$ and $t$

• Nth order statistics
  
  – For a complete characterization of a stochastic process, we require the knowledge of all nth order statistics
    • **1st order statistics**: Stationary distribution, expectation (at time $t$)
    • **2nd order statistics**: Covariance (auto covariance), Correlation
Stochastic process (contd.)

• **Stationary process**
  - all \( n \) order statistics are translational invariant

• **Stationary in wide sense**
  - Only 1\(^{st}\) and 2\(^{nd}\) order (mean and covariance respectively) statistics are translation invariant

• **Process of stationary increments**
  \( X_{t+T} - X_t \) is a stationary process for all \( T \)

• **Ergodic process**
  the whole statistics (usually 1\(^{st}\) and 2\(^{nd}\) order are sufficient) of the process can be determined from a single (infinitely long) realization
Poisson process

- Is used to describe arrival process of customer/call (the population is considered infinite)
- A counter process $N(t_1, t_2)$: describes number of arrivals in the interval $(t_1, t_2]$

- Definition
  - A pure birth process (for a infinitesimal time interval, only one arrival may occur)
  - $N(t)$ obeys Poisson($\lambda t$) distribution: where $\lambda$ is arrival intensity (mean arrival rate, probability of arrival per unit time)
  - Interarrival times are independent and obeys exponential distribution
- Memoryless property of exponential distribution
Group work

• Hitchhiker waiting for a car
  – Car arrivals can be modeled as Poisson process
  – Mean interval between the cars is 10 min.
  – If hitchhiker arrives to the roadside at random instant of time
  – What will be mean waiting time?
  – What will be the mean waiting time for the same hitchhiker if he is standing on a bus station that arrives after every 10 min. ?

• Hitchhiker paradox
Properties of Poisson process

• **Superposition**: the superposition of two Poisson processes with intensities $\lambda_1$ and $\lambda_2$ is a Poisson process with intensity $\lambda_1 + \lambda_2$

• **Random selection**: For a Poisson process with intensity $\lambda$, a random selection of arriving process with probability $p$ (independent of others) results in a Poisson process with intensity $p\lambda$

• **Random split**: A random split of a Poisson process $(\lambda)$ with probability $p_i$ ($\sum p_i$) results in Poisson sub-processes of intensities $\lambda p_i$

• **Poisson arrival see time averages (PASTA)**: customers with Poisson arrivals see the system as if they came into the system at a random instant of time

• **Palm theorem**: superposition of renewal processes tend to a Poisson process
  – Renewal process – independent, identically distributed (iid) inter-arrival times
Markov process

• Stochastic process with the property
  – \( P(X(t_{n+1}) = j \mid X(t_n) = i, X(t_{n-1}) = l, \ldots, X(t_0) = m) = P(X(t_{n+1}) = j \mid X(t_n) = i) \)
  – The current state \( X(t_{n+1}) \) doesn’t depend on future or previous state (future path of the Markov process only depends on the current state not how it is reached)

• Homogenous Markov process
  – \( P( X(t_{n+1}) = j \mid X(t_n) = i ) = P( X(t+(t_{n+1}-t_n)) = j \mid X(t)=i ) = p_{ij}(t_{n+1}-t_n) \)
  – Probability values will always be the same at \( \Delta t \) time interval

• Markov chain: if state space is discrete a Markov process can be represented by graph
  – States: nodes
  – State changes: edges
Discrete-time Markov chains

• Discrete-time Markov-chain: the time of state change is discrete as well (discrete time, discrete space stochastic process)
  – State transition probability: the probability of moving from state i to state j in one time unit.

• We will not consider them in this course!!!!
Continuous-time Markov chains (homogeneous case)

- Continuous time, discrete space stochastic process, with Markov property
- State transition can happen at any point in time
- The time spent in a state has to be exponential to ensure Markov property
- The Markov chain is characterized by the state transition matrix $Q$ – the probability of $i$ to $j$ state transition in $\Delta t$ time is:

$$q_{ij} = \lim_{\Delta t \to 0} \frac{P(X(t+\Delta t)=j|X(t)=i), t=j}{\Delta t}$$

$$q_{ii} = -\sum_{i \neq j} q_{ij}$$

$$q_i = \sum_{i \neq j} q_{ij}$$ - time spent in state $i$ (holding time) : $\exp(q_i)$

- Transition rate matrix:

$$Q = \begin{bmatrix}
q_{00} & \cdots & q_{0M} \\
\vdots & \ddots & \vdots \\
q_{M0} & \cdots & q_{MM}
\end{bmatrix}$$

$$Q = \begin{bmatrix}
-4 & 4 \\
6 & -6
\end{bmatrix}$$
Continuous-time Markov chains (homogeneous case)

- Transition rate matrix:

\[ Q = \begin{bmatrix}
q_{00} & \cdots & q_{0M} \\
\vdots & \ddots & \vdots \\
q_{M0} & \cdots & q_{MM}
\end{bmatrix} \]

- \( q_{01} = 12 \)
- \( q_{10} = 10 \)

\[ Q = \begin{bmatrix}
-12 & 12 \\
10 & -10
\end{bmatrix} \]
Transient solution

- The transient - time dependent – state probability distribution
- \( \mathbf{p}(t) = \{p_1(t), p_2(t), p_3(t), \ldots\} \) – probability of being in state \( i \) at time \( t \), given \( \mathbf{p}(0) \).

\[
p_i(t + \Delta t) = p_i(t) - p_i(t) \sum_{j \neq i} q_{ij} \Delta t + \sum_{j \neq i} p_j(t)q_{ji} \Delta t + o(\Delta t), \quad \lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0
\]

leaves the state \( i \) \quad arrives to the state \( j \)

\[
\mathbf{p}(t + \Delta t) = \mathbf{p}(t)(I + Q\Delta t) + o(\Delta t)
\]

\[
\frac{\mathbf{p}(t + \Delta t) - \mathbf{p}(t)}{\Delta t} \to \frac{d\mathbf{p}(t)}{dt} = \mathbf{p}(t)Q
\]

\[
\mathbf{p}(t) = \mathbf{p}(0) \cdot e^{Qt}
\]

Transient solution
Example - Transient solution

\[
\frac{dp(t)}{dt} = p(t)Q, \quad p(t) = \{p_0(t), p_1(t)\}
\]

\[
p_0(t)' = p_0(t)q_{00} + p_1(t)q_{01} = -4p_0(t) + 6p_1(t)
\]

\[
p_1(t)' = p_0(t)q_{10} + p_1(t)q_{11} = 4p_0(t) - 6p_1(t)
\]

or: \(p_0(t) + p_1(t) = 1\)
Stationary solution (steady state)

- Def: stationary state probability distribution (stationary solution)
  - \( p = \lim_{t \to \infty} p(t) \) exists
  - \( p \) is independent from \( p(0) \)
- Stationary solution exist, if
  - The Markov chain is irreducible
    (there is a path between any two states)
  - \( pQ = p' = 0, \quad p \times 1 = 1 \) has positive solution
- Finite state, irreducible Markov chains always have stationary solution.

\[
\begin{bmatrix} p_0, p_1 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 6 & -6 \end{bmatrix} = [0, 0], \quad p_0 + p_1 = 1
\]

\[
p_0 = 0.6, \quad p_1 = 0.4
\]
Ergodicity

• A Markov chain is ergodic if it has a stationary solution

• Ergodic theorem: if a process ergodic, then the statistics of the process can be determined from a single (infinitely long) realization
  – Consequence: stationary state probability
    • Probability that the process is in state $i$ at a given point of time
    • Part of the time a single realization spends in state $i$
Balance equations

• Method to find stationary solution \( pQ = 0 \)
• Global balance equation
  – Conditions
    • In equilibrium (for stationary solution)
    • Flow in = Flow out

• Group work
  – Global balance equations for
    • State 1 & 2
    • Dashed circle
Balance equations

• Local balance equations
  – The flow from one part of the chain should be equal to flow back from the other part (in static state)

• Calculating the steady state distribution
  – With matrix eq. \( \mathbf{pQ} = 0, \mathbf{px}1 = 1 \)
  – With balance eq. (local/global)
    • Calculate M states
    • M-1 balance equations and \( \sum \mathbf{p} = 1 \)
Balance equations

• Calculating the steady state distribution
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