Example: Existence of Superman

If Superman were able and willing to prevent evil, then he would so. If Superman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

Is this argument valid?
Definitions

By an argument, we mean a sequence of statements that ends with a conclusion.

The conclusion is the last statement of the argument.

The premises are the statements of the argument preceding the conclusion.

By a valid argument, we mean that the conclusion must follow from the truth of the premises.

Rule of Inference

Some tautologies are rules of inference. The general form of a rule of inference is

\[(p_1 \land p_2 \land \cdots \land p_n) \rightarrow c\]

where

- \(p_i\) are the premises

and

- \(c\) is the conclusion.

Notation

A rule of inference is written as

\[
p_1 \quad p_2 \quad \vdots \quad p_n \quad \therefore \quad c
\]

where the symbol \(\therefore\) denotes "therefore". Using this notation, the hypotheses are written in a column, followed by a horizontal bar, followed by a line that begins with the therefore symbol and ends with the conclusion.
modus ponens

The rule of inference

\[
p \rightarrow q
\]

\[
p
\]

\[
\therefore q
\]

is denoted the law of detachment or modus ponens (Latin for mode that affirms). If a conditional statement and the hypothesis of the conditional statement are both true, therefore the conclusion must also be true.

The basis of the modus ponens is the tautology

\[
((p \rightarrow q) \land p) \rightarrow q.
\]

Example of modus ponens

If it rains, then it is cloudy.
It rains.
Therefore, it is cloudy.

\[r \rightarrow c\]

\[r\]

\[\therefore c\]
modus tollens

The rule of inference

\[ p \rightarrow q \]
\[ \neg q \]
\[ \therefore \neg p \]

is denoted the modus tollens (Latin for mode that denies). This rule of inference is based on the contrapositive. The basis of the modus ponens is the tautology

\[ ((p \rightarrow q) \land \neg q) \rightarrow \neg p. \]

Example of modus tollens

If it rains, then it is cloudy.
It is not cloudy.
Therefore, it is not the case that it rains.

\[ r \] is the proposition “it rains.”
\[ c \] is the proposition “it is cloudy.”

\[ r \rightarrow c \]
\[ \neg c \]
\[ \therefore \neg r \]

The Addition

The rule of inference

\[ p \]
\[ \therefore p \lor q \]

is the rule of addition.

This rule comes from the tautology

\[ p \rightarrow (p \lor q). \]
The Simplification

The rule of inference

\[
\begin{align*}
p \land q \\
\therefore \ p
\end{align*}
\]

is the rule of *simplification*. This rule comes from the tautology

\[
(p \land q) \rightarrow p.
\]

The Hypothetical Syllogism

The rule of inference

\[
\begin{align*}
p \rightarrow q \\
q \rightarrow r \\
\therefore \ p \rightarrow r
\end{align*}
\]

is the rule of *hypothetical syllogism* (syllogism means “argument made of three propositions where the last one, the conclusion, is necessarily true if the two firsts, the hypotheses, are true”). This rule comes from the tautology

\[
(((p \rightarrow q) \land (q \rightarrow r)) \rightarrow (p \rightarrow r).
\]

The Disjunctive Syllogism

The rule of inference

\[
\begin{align*}
p \lor q \\
\neg p \\
\therefore \ q
\end{align*}
\]

is the rule of *disjunctive syllogism*. This rule comes from the tautology

\[
((p \lor q) \land \neg p) \rightarrow q.
\]

The Conjunction

The rule of inference

\[
\begin{align*}
p \\
q \\
\therefore \ p \land q
\end{align*}
\]

is the rule of *conjunction*. This rule comes from the tautology

\[
((p \land q) \rightarrow (p \land q).
\]
The Resolution

The rule of inference

\[
\begin{align*}
p \lor q \\
\neg p \lor r \\
\hline
\therefore q \lor r
\end{align*}
\]

is the rule of **resolution**.

This rule comes from the tautology

\[(p \lor q) \land (\neg p \lor r) \rightarrow (q \lor r).\]

Outline

Rules of Inference
- Motivation
- Definitions
- Rules of Inference
- Fallacies
- Using Rules of Inference to Build Arguments
- Rules of Inference and Quantifiers

Fallacies

**Fallacies** are incorrect arguments.

Fallacies resemble rules of inference but are based on **contingencies** rather than tautologies.

The Fallacy of Affirming the Conclusion

The wrong "rule of inference"

\[
\begin{align*}
p \rightarrow q \\
q \\
\hline
\therefore p
\end{align*}
\]

is denoted the **fallacy of affirming the conclusion**.

The basis of this fallacy is the contingency

\[(q \land (p \rightarrow q)) \rightarrow p\]

that is a misuse of the **modus ponens** and is not a tautology.
Fallacy of Affirming the Conclusion

If it rains, then it is cloudy.
It is cloudy.
Therefore, it rains (wrong).

\( r \) is the proposition “it rains.”
\( c \) is the proposition “it is cloudy.”

\[
\begin{align*}
  r & \implies c \\
  c & \\
  \therefore r \ (\text{wrong})
\end{align*}
\]
Example of the Fallacy of Denying the Hypothesis

If it rains, then it is cloudy.
It is not the case that it rains.
Therefore, it is not cloudy (wrong).

$r$ is the proposition “it rains.”
$c$ is the proposition “it is cloudy.”

\[ r \rightarrow c 
\]
\[ \neg r 
\]
\[ \therefore \neg c \text{ (wrong)} \]

Example: Existence of Superman

If Superman were able and willing to prevent evil, then he would so. If Superman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

• $w$ is “Superman is willing to prevent evil”
• $a$ is “Superman is able to prevent evil”
• $i$ is “Superman is impotent”
• $m$ is “Superman is malevolent”
• $p$ is “Superman prevents evil”
• $x$ is “Superman exists”

Example: Existence of Superman

If Superman were able and willing to prevent evil, then he would so. If Superman were unable to prevent evil, then he would be impotent; if he were unwilling to prevent evil, then he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

• $h1. \ (a \land w) \rightarrow p$
• $h2. \ \neg a \rightarrow i$
• $h3. \ \neg w \rightarrow m$
• $h4. \ \neg p$
• $h5. \ x \rightarrow \neg i$
• $h6. \ x \rightarrow \neg m$
Example: Existence of Superman

Argument:
1. \( \neg i \to a \) contrapositive of \( h_2 \).
2. \( x \to a \) \( h_5 \) and step 1 with hyp. syll.
3. \( \neg m \to w \) contrapositive of \( h_3 \).
4. \( x \to w \) \( h_6 \) ans step 3 with hyp. syll.
5. \( x \to (a \land w) \) Step 2 and 4 with conjunction.
6. \( x \to p \) Step 5 and \( h_1 \) with hyp. syll.
7. \( \neg x \) Step 6 and \( h_4 \) with modus tollens.

Q.E.D.

Outline

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- Motivation
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- Rules of Inference and Quantifiers

Rules of Inference and Quantifiers

There are four rules of inference for quantifiers:
- Universal instantiation (UI),
- Universal generalization (UG),
- Existential instantiation (EI),
- Existential generalization (EG).

Universal Instantiation

\[
\exists x P(x) \\
\therefore P(c)
\]

If a propositional function is true for all element \( x \) of the universe of discourse, then it is true for a particular element \( c \) of the universe of discourse.
Universal Instantiation and *modus ponens*

The universal instantiation and the *modus ponens* are used together to form the **universal modus ponens**. Example: *All humans have two legs. John Smith is a human. Therefore, John Smith has two legs.*

- $H(x)$ is “$x$ is a human.”
- $L(x)$ is “$x$ has two legs.”
- $j$ is John Smith, an element of the universe of discourse.

1. $\forall x (H(x) \rightarrow L(x))$ Premise.
2. $H(j) \rightarrow L(j)$ Universal instantiation from 1.
3. $H(j)$ Premise.

∴ $L(j)$ **Modus ponens** from 2. et 3.

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**Universal Generalization**

$P(c) \quad \text{for an arbitrary } c$

\[ \therefore \forall x P(x) \]

We must first define the universe of discourse. Then, we must show that $P(c)$ is true for an arbitrary, and not a specific, element $c$ of the universe of discourse. We have no control over $c$ and we can not make any other assumptions about $c$ other than it comes from the domain of discourse. The error of adding unwarranted assumptions about the arbitrary element $c$ is common and is an incorrect reasoning.

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**Existential Instantiation**

$\exists x P(x) \quad \therefore \quad P(c) \quad \text{for some element } c$

The existential instantiation is the rule that allows us to conclude that there is an element $c$ in the universe of discourse for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We can not select an arbitrary value of $c$ here, but rather it must be a $c$ for which $P(c)$ is true.

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**Existential Generalization**

$P(c) \quad \text{for some element } c$

\[ \therefore \exists x P(x) \]

If we know one element $c$ in the universe of discourse for which $P(c)$ is true, therefore we know that $\exists x P(x)$ is true.